Notes on Étale Cohomology.

Fall 2015.

Étale Cohomology.

fpqc descent (affine case).

Defn & Prop.
$A \to B$, TFAE.

① $A\text{-mod} \xrightarrow{\otimes B} B\text{-mod}$ is faithful and exact

② B is flat and $\forall M \neq 0$ $B \otimes_A M \neq 0$.

③ $0 \to M_1 \to M_2 \to M_3 \to 0$ exact iff $B \otimes_A -$ is exact.

④ B is flat and $\operatorname{Spec} B \to \operatorname{Spec} A$ is surj.

and we call ~~A~~ B faithfully flat $/A$.

Lemma. (permanence of properties of modules)
Let $f: A \to B$ faithfully flat, M an A-mod., then $B \otimes_A M$ satisfying following iff M sat. them:

① f.g. ② f.p. ③ flat ④ l.f. of rank n.

Lemma. (permanence of properties of morphisms)
$A \to B$ faithfully flat
Let $S = \operatorname{Spec} A$, $T = \operatorname{Spec} B$. let $f: \begin{tikzcd}[column sep=small] X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$ be a morphism of A-schemes, $f_T: X_T \to Y_T$, then f_T is … iff f is …, where … could be

① l.f.t. ② l.f.p. ③ flat. ④ formally unramified.
⑤ étale.

Defn.
Let $A \to B$. A descent datum for f is a pair (N, φ) where $\varphi: N \otimes_A B \to B \otimes_A N$ is a $B \otimes_A B$ isom., s.t.

$$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}"'] & \arrow[d, phantom, "\circlearrowright"] & B \otimes_A B \otimes_A N \\ & B \otimes_A N \otimes_A B \arrow[ur, "\varphi_{12}"'] & \end{tikzcd}$$

A morphism $(N,\varphi) \xrightarrow{f} (N', f\varphi')$ is a B-morph $N \xrightarrow{f} N'$ s.t.

$$\begin{tikzcd} N\otimes_A B \arrow[r, "\varphi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & B\otimes_A N \arrow[d, "f"] \\ N'\otimes_A B \arrow[r, "\varphi'"] & B\otimes N' \end{tikzcd}$$

e.g. for M an A-mod. the canonical descent datum is $(M\otimes_A B, \mathrm{can})$ where $(M\otimes_A B)\otimes_A B \longrightarrow B\otimes_A(M\otimes_A B)$.

$$m\otimes b\otimes c \longmapsto b\otimes m\otimes c.$$

This gives $F: A\text{-mod} \longrightarrow f\text{-descent}$

Thm* (fpqc descent) If f is f.f., then F is an eq.

Rmk as f is f.f, F is faithfully exact.

Rmk it says, if we have $\mathcal{U} \xrightarrow{f} X$, fpqc. $\pi_{1,2}: \mathcal{U}\times_X \mathcal{U} \to \mathcal{U}$. qcoh-$\mathcal{F}$ sheaf on $\mathcal{U}$, $\pi_1^*(\mathcal{F}) \xrightarrow{\sim} \pi_2^*(\mathcal{F})$ satisfying cocycle condition on $\mathcal{U}\times_X\mathcal{U}\times_X\mathcal{U}$; glue $\mathcal{F}$ to a q-coh. sheaf on X.

Grothendieck pretopology.

Defn Let $\mathcal{C}$ be a category. A Grothendieck pretop. on $\mathcal{C}$ is a collection $\mathrm{Cov}(\mathcal{C})$ of coverings $\{\mathcal{U}_i \to \mathcal{U}\}_{i\in I}$ s.t.

(0) if $\mathcal{U}_i \to \mathcal{U} \in \mathcal{U}$, $V \to \mathcal{U}$ arbitrary, then $\mathcal{U}_i \times_{\mathcal{U}} V$ exists

(1) If $\{\mathcal{U}_i \to \mathcal{U}\} \in \mathrm{Cov}(\mathcal{C})$ and $V \to \mathcal{U}$ arbitrary, then $\{\mathcal{U}_i\times_{\mathcal{U}} V \to V\} \in \mathrm{Cov}(\mathcal{C})$.

(2) If $\{\mathcal{U}_i \to \mathcal{U}\}$, $\{\mathcal{U}_{ik} \to \mathcal{U}_i\} \in \mathrm{Cov}(\mathcal{C})$, then $\{\mathcal{U}_{ik} \to \mathcal{U}\} \in \mathrm{Cov}(\mathcal{C})$.

(3) If $V \xrightarrow{\sim} \mathcal{U}$, then $\{V \xrightarrow{\sim} \mathcal{U}\} \in \mathrm{Cov}(\mathcal{C})$.

Defn. A site is a cat $\mathcal{C}$ w/ a pretop. on it.

e.g. (small) étale site ~~(small)~~ Let X be a scheme, let $\mathcal{C} = \text{ét}/X$, and let coverings be $\mathrm{Cov}(\mathcal{C}) = \{\{\mathcal{U}_i \to \mathcal{U} \mid \bigcup \text{image} = \mathcal{U}\}\}$

e.g. (large) fppf site X scheme, let $\mathcal{C} = Sch/X$. Let coverings be $\mathrm{Cov}(\mathcal{C}) = \{\{\mathcal{U}_i \to \mathcal{U} \mid \text{jointly surj. } \mathcal{U}_i \to \mathcal{U}$ ~~jointly~~ flat of finite presentation$\}\}$

Defn. A presheaf is a functor $\mathcal{C}^{op} \to \mathcal{D}$.

Defn. A presheaf $\mathcal{F}$ is a sheaf if $\forall \{\mathcal{U}_i \to \mathcal{U}\}$

$$\mathcal{F}(\mathcal{U}) \to \prod_{i\in I} \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(\mathcal{U}_i\times_{\mathcal{U}}\mathcal{U}_j)$$

is an equalizer. If we only know $\mathcal{F}(\mathcal{U}) \hookrightarrow \prod_i \mathcal{F}(\mathcal{U}_i)$, then $\mathcal{F}$ is separated.

Defn. $\underline{PSh_{\mathcal{D}}}(\mathcal{C})$ $\left(\underline{Sh_{\mathcal{D}}}(\mathcal{C})\right)$ cat. of presheaves (sheaves).

Lemma. Suppose $\mathcal{D}$ is (co)complete, then so is $\underline{PSh_{\mathcal{D}}}(\mathcal{C})$ and (co)lim's are "pointwise"

$$\left((\mathrm{co})\lim F_i\right)(U) = (\mathrm{co})\lim\left(F_i(U)\right).$$

e.g. $F \xrightarrow{f} G$ in $\underline{PSh_{\mathcal{D}}}(\mathcal{C})$ then f is monic (epic, isom.) if each $F(U) \to G(U)$ is monic (epic, isom.).

Because $B \to C$ monic $\Leftrightarrow$
$$\begin{tikzcd} B \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\lrcorner"] & C \end{tikzcd}$$

$B \to C$ epic $\Leftrightarrow$
$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ C \arrow[r] & C \end{tikzcd}$$

Čech Coh: Let F be a presheaf. Let $\mathcal{U} = \{U_i \to U\}$ be a covering, then Čech cplx of F w/ resp. to $\mathcal{U}$ is

$$\check{C}^p(\mathcal{U}, F) = \prod_{(i_0, \dots, i_p) \in I^{p+1}} F(U_{i_0} \times_U \cdots \times_U U_{i_p}).$$

$$\check{C}^p \xrightarrow{\delta} \check{C}^{p+1}, \quad s_i \xmapsto{\delta} \sum_{j=0}^{p} (-1)^j \left(s_{i_0 \cdots \hat{i_j} \cdots i_{p+1}}\right)\Big|_{(i_0 \cdots i_{p+1})}.$$

$\delta^2 = 0$.

The Čech coh. of F w.r.t. $\mathcal{U}$ is $\check{H}^p(\mathcal{U}, F) = H^p(\check{C}^\bullet(\mathcal{U}, F))$.

Defn. Let $\mathcal{U} = \{U_i \to U\}_i$, $\mathcal{V} = \{V_j \to U\}_j$ covers, $\mathcal{V}$ is a refinement of $\mathcal{U}$ if $\exists\, \alpha : J \to I$, and
$$\begin{tikzcd} V_j \arrow[r, "\eta_j"] \arrow[dr] & U_{\alpha(j)} \arrow[d] \\ & U \end{tikzcd}$$
The $(\alpha, (\eta_j))$ is called a refining morphism.

$(\alpha, (\eta_j))$ defines $\check{C}^\bullet(\mathcal{U}, F) \longrightarrow \check{C}^\bullet(\mathcal{V}, F)$ induced by
$$V_{j_0 \cdots j_p} \longrightarrow U_{\alpha(j_0) \cdots \alpha(j_p)}. \qquad \mathcal{V} \sqsubset \mathcal{U} \text{ (refinement)}$$

Lemma. Let $(\alpha, (\eta_j))$, $(\beta, (\theta_j))$ be 2 refining $\mathcal{V} \sqsubset \mathcal{U}$, they induce same map on coh.

pf. construct homotopy: $\check{C}^{p+1}(\mathcal{U}, F) \longrightarrow \check{C}^p(\mathcal{V}, F)$.
$$(s_i) \longmapsto \sum_{k=0}^{p} (-1)^k \eta_{j_0} \cdots \eta_{j_k} \theta_{j_{k+1}} \cdots \theta_{j_p} \left(s_{\alpha(j_0) \cdots \alpha(j_k) \beta(j_{k+1}) \cdots \beta(j_p)}\right).$$

Rmk Thus it suffices to look at the (possibly large) set
$$J_U = \mathrm{Cov}(U)/\equiv, \text{ where } \mathcal{U} \equiv \mathcal{V} \text{ iff } \mathcal{U} \sqsubset \mathcal{V} \sqsubset \mathcal{U}.$$
Then J_U is partially ordered by $\sqsubset$, and directed, for $\mathcal{U}, \mathcal{V}$, $\exists\ \mathcal{U} \times_U \mathcal{V}$.

Defn. Čech coh. of F on U is $\operatorname{colim}_{\mathcal{U} \in J_U} \check{H}^p(\mathcal{U}, F)$

Warning ⚠ J_U could not be a set!

Sheafification for $V \to \mathcal{U}$, $\exists\ J_{\mathcal{U}} \to J_V$, gives a map

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^p(V, \mathcal{F}).$$

Thus, the $\check{H}^p(-, \mathcal{F})$ gives a presheaf: $\check{\mathcal{H}}^p(\mathcal{F})$.

trivial cover $\{U \to U\}$ gives $\mathcal{F}(U) \longrightarrow \check{\mathcal{H}}^0(\mathcal{F})(U)$

which gives $\mathcal{F} \longrightarrow \check{\mathcal{H}}^0 := \mathcal{F}^+$, functorial in $\mathcal{F}$.

If $\mathcal{F}$ is separated, then $\mathcal{F} \to \mathcal{F}^+$ is inj.

If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \xrightarrow{\sim} \mathcal{F}^+$.

Thm Let $\mathcal{F}$ be a presheaf.

(1) $\mathcal{F}^+$ is ~~separable~~ separated

(2) $\mathcal{F}$ is separated $\Rightarrow$ $\mathcal{F}^+$ is a sheaf.

pf. (1) let $s \in \mathcal{F}^+(U)$, s.t. for $\{U_i \to U\}$, $s|_{U_i} = 0$ in $\mathcal{F}^+(U_i) = \operatorname{colim}_{\{U_{ik} \to U_i\}} \check{H}^0(\mathcal{U}_i, \mathcal{F})$. So $\exists\ \{U_{ik} \to U_i\}$, s.t. $s|_{U_{ik}} = 0$. Now pass to $\{U_{ik} \to U_i \to U\}$, $s|_{U_{ik}}$ will be given by 0 in $\mathcal{F}(U_{ik})$, so $s = 0$ in $\mathcal{F}^+(U)$.

(2) $\rho: \mathcal{F} \to \mathcal{F}^+$ is injective.

Let $s_i \in \mathcal{F}^+(U_i)$, s.t. $s_i|_{U_i \times U_j} = s_j|_{U_i \times U_j}$

Choose $\{U_{ik} \to U_i\}$, $s_{ik} \in \mathcal{F}(U_{ik})$, s.t. $s_i|_{U_{ik}} = \rho(s_{ik})$.

Then $\rho(s_{ik})|_{U_{ik} \times U_{i'k'}} = s_i|_{U_{ik} \times U_{i'k'}} = s_{i'}|_{U_{ik} \times U_{i'k'}} = \rho(s_{i'k'})$

By injectivity of ρ, we get $s_{ik}|_{U_{ik} \times U_{i'k'}} = s_{i'k'}|_{U_{ik} \times U_{i'k'}}$. Thus

$s = (s_{ik})_{i,k} \in \check{H}^0(W, \mathcal{F})$, where $W = \{U_{ik} \to U_i \to U\}$.

Cor. Let $\mathcal{F}$ presheaf. (1) $\mathcal{F}^{++}$ is a sheaf.

(2) $\mathcal{F}$ is sep. iff $\mathcal{F} \hookrightarrow \mathcal{F}^+$

(3) $\mathcal{F}$ is a sheaf iff $\mathcal{F} \xrightarrow{\sim} \mathcal{F}^+$.

Lemma. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a sheaf.

If $g: \mathcal{F}^+ \to \mathcal{G}$, s.t. $g\rho = 0$, then $g = 0$

pf: given $s \in \mathcal{F}^+(U)$, $s|_{U_i} = \rho(s_i)$, hence $g(s)|_{U_i} = 0$.

Now $\mathcal{G}$ is a sheaf, so $g(s) = 0$.

Thm. $\mathcal{F}$ presheaf, $\mathcal{G}$ sheaf. $\operatorname{Hom}_{Sh}(\mathcal{F}^{++}, \mathcal{G}) = \operatorname{Hom}_{PSh}(\mathcal{F}, \mathcal{G})$

pf.
$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\rho"] \arrow[d] & \mathcal{F}^+ \arrow[r, "\rho"] \arrow[d] & \mathcal{F}^{++} \arrow[d] \\
\mathcal{G} \arrow[r, "\sim"] & \mathcal{G}^+ \arrow[r, "\sim"] & \mathcal{G}^{++}
\end{tikzcd}$$
shows surjectivity. (of $\longrightarrow$)

Now suppose $f_1, f_2: \mathcal{F}^{++} \to \mathcal{G}$, s.t. $f_1\rho\rho = f_2\rho\rho$. by Lemma before, $f_1 = f_2$.

Cor. $PSh(\mathcal{C}) \underset{id(+)}{\overset{++}{\rightleftarrows}} Sh(\mathcal{C})$, $++$ is left adjoint of id.

Cor. Let $D: \mathcal{J} \to \underline{Sh}(\mathcal{C})$ be a diagram, then the colim $\varinjlim Sh(\mathcal{C})$ is given by sheafification of pointwise colim

pf. sheafification is left adjoint, so keeps colims, and is identity on the subcat. of sheaves.

Rmk. limits are just limits in presheaf: if all $\mathcal{F}_i$ are sheaves, then so is $\lim \mathcal{F}_i$.

Cor. Sheaves is (co) complete.

Thm*. $\underline{Sh}_{\mathcal{D}}(\mathcal{C})$ is an abelian cat. (need $\mathcal{J}_U$ be small)

Lemma. sheafification is exact and commute w/ restriction to a subcat $\mathcal{C}' \subseteq \mathcal{C}$.

pf. right exact: left adjoint (∃) (✓).

left exact: by hand (✓).

Change of sites.

Setup: $u: \mathcal{C} \to \mathcal{D}$, be a functor (of sites). assume at some pt: $(*)$ $\mathcal{C}$ has and u preserves: fibred product and terminal objects (hence all finite limits).

Ex. $X \xrightarrow{f} Y$, continuous, $u: \underline{Top}(Y) \to \underline{Top}(X)$, $U \longmapsto f^{-1}(U)$.

Lemma Functor $u^s: \underline{PSh}(\mathcal{D}) \to \underline{PSh}(\mathcal{C})$ is exact

$\mathcal{F} \longmapsto (U \longmapsto \mathcal{F}(u(U)))$

Defn. comma cat. $(A \downarrow u)$ where $A \in Ob(\mathcal{D})$.

ob: $\begin{pmatrix} A \\ \downarrow f \\ u(U) \end{pmatrix}$ where $U \in Ob(\mathcal{C})$, $f \in \mathcal{D}(A, u(U))$.

mor:
$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ u(U) \arrow[rr, "u(\varphi)"'] & & u(V) \end{tikzcd}$$

Defn. The functor $u_p: \underline{PSh}(\mathcal{C}) \longrightarrow \underline{PSh}(\mathcal{D})$

$\mathcal{F} \longmapsto \left(A \longmapsto \operatorname{colim}_{(U,f) \in (A \downarrow u)^{op}} \mathcal{F}(U)\right)$

Thm*. For $\mathcal{F} \in \underline{PSh}(\mathcal{C})$, $\mathcal{G} \in \underline{PSh}(\mathcal{D})$, we have

$\mathrm{Hom}_{\underline{PSh}(\mathcal{D})}(u_p(\mathcal{F}), \mathcal{G}) = \mathrm{Hom}_{\underline{PSh}(\mathcal{C})}(\mathcal{F}, u^p(\mathcal{G}))$.

Lemma. If $(*)$ holds, then u_p is exact.

pf. $(A \downarrow u)$ is cofiltered, and $(A \downarrow u)^{op}$ is filtered, and filtered limits are exact.

Defn. (for us) a functor $u: \mathcal{C} \to \mathcal{D}$ of sites is continuous if $(*)$ holds and for each covering $\{U_i \to U\}$ in $\mathcal{C}$, $\{u(U_i) \to u(U)\}$ is a covering in $\mathcal{D}$.

e.g. for $f: X \to Y$ schemes, $\acute{E}t/Y \longrightarrow \acute{E}t/X$ is continuous

similarly for the big fppf, étale, ... small Zariski site.

e.g. We get functors. $X_{Zar} \to X_{et} \to X_{fppf}$

Lemma. Let $u: \mathcal{C} \to \mathcal{D}$ be continuous. If F is a sheaf on $\mathcal{D}$, then $u^p F$ is a sheaf on $\mathcal{C}$.
pf: sheaf condition of $u^p F$ on $\{U_i \to U\}$ is just sheaf condition of F...

Defn. functor: $\underline{Sh}(\mathcal{D}) \to \underline{Sh}(\mathcal{C})$ is denoted u^s.

Defn. $\underline{Sh}(\mathcal{C}) \hookrightarrow \underline{PSh}(\mathcal{C}) \xrightarrow{u_p} \underline{PSh}(\mathcal{D}) \xrightarrow{(-)^{++}} \underline{Sh}(\mathcal{D})$ is denoted u_s.

Thm. $\mathrm{Hom}_{\underline{Sh}(\mathcal{D})}(u_s F, \mathcal{G}) = \mathrm{Hom}_{\underline{Sh}(\mathcal{C})}(F, u^s \mathcal{G})$.

Lemma. (If (*)) then u_s is exact, as each step is exact.

Defn. A <u>morphism of sites</u> $f: \mathcal{D} \to \mathcal{C}$ is a continuous functor $u: \mathcal{C} \to \mathcal{D}$ (s.t. u_s is exact).

Ex. For $f: X \to Y$ morphism of schemes, we get $f: X_{et} \to Y_{et}$
$X_{fppf} \to X_{et} \to X_{Zar}$.

Defn. If $f: \mathcal{D} \to \mathcal{C}$ is a morphism of sites, we write $f^* = u_s$, $f_* = u^s$.

Thm. $f: \mathcal{D} \to \mathcal{C}$ morphism of sites, $f^* \dashv f_*$ and f^* is exact.
Rmk. for schemes, we call f^* here by f^{-1}.

Cohomology.

Thm If $\mathcal{C}$ is small, $\underline{Sh}(\mathcal{C})$ is a Grothendieck abelian cat.
In particular, it has enough inj.'s.

Defn.
- derived functor of $\Gamma(U,-)$ are denoted $H^i(U,-)$
- —"— $\underline{Sh}(\mathcal{C}) \to \underline{PSh}(\mathcal{C})$ are denoted $\mathcal{H}^i$,
  $\mathcal{H}^i(F)(U) = H^i(U, F)$.
- If $f: \mathcal{D} \to \mathcal{C}$ is a morphism, derived functors of f_* are denoted $R^i f_*: \underline{Sh}(\mathcal{D}) \to \underline{Sh}(\mathcal{C})$

Thm (?) functors $\check{\mathcal{H}}^i: \underline{PSh}(\mathcal{C}) \to \underline{PSh}(\mathcal{C})$ form a universal δ-functor
pf. prove they are effaceable

Prop. Let $G: \mathcal{B} \to \mathcal{A}$ have an exact left adjoint, then G preserves inj.

Ex.
- $\underline{Sh}(\mathcal{C}) \xrightarrow{\mathcal{H}^0} \underline{PSh}(\mathcal{C})$.
- For $f: \mathcal{D} \to \mathcal{C}$ morph of sites, get $f_*: \underline{Sh}(\mathcal{D}) \to \underline{Sh}(\mathcal{C})$

Cor. (Spectral Sequence). $f: \mathcal{D} \to \mathcal{C}$ morphism of sites, then.
$E_2^{pq} = H^p(U, R^q f_*(-)) \Rightarrow H^{p+q}(u(U), -)$.
$\mathcal{E}_2^{pq} = \mathcal{H}^p(R^q f_*(-)) \Rightarrow \mathcal{H}^{p+q}(-)$.
$E_2^{pq} = H^p(U, \mathcal{H}^q(-)) \Rightarrow H^{p+q}(U, -)$.
$\mathcal{E}_2^{pq} = \check{\mathcal{H}}^p(\mathcal{H}^q(-)) \Rightarrow \mathcal{H}^{p+q}(-)$. where $(-)$ is a sheaf

Lemma. $\check{H}^0(\mathcal{H}^q(-))=0$ for $q>0$ ($\mathcal{H}^q$'s are ~~presheaf~~ presheaves w/ 0 stalk)

pf. Recall $\check{\mathcal{H}}^{0\circ} \hookrightarrow (\ )^{++}$, suffices to prove $\mathcal{H}^q(F)^{++}=0$

Let $J^\cdot$ be an inj. resolution. Then $\mathcal{H}^q(F)=H^q(\mathcal{H}^0(J^\cdot))$.

Now $(\ )^{++}$ is exact, so $H^q(\mathcal{H}^0(J^\cdot))^{++}=H^q(\mathcal{H}^0(J^\cdot)^{++})$

$=H^q(J^\cdot)=0$.

Cor. $\mathcal{H}^0=\check{\mathcal{H}}^0$, $\mathcal{H}^1=\check{\mathcal{H}}^1$ ($H^0(U,-)=\check{H}^0(U,-)$)

& $H^1(U,-)=\check{H}^1(U,-)$ on sheaves.

Thm If X is q.proj. over an affine, then $\check{H}^i_{et}=H^i_{et}$.

Lemma Let F be a presheaf on X_{et}. Then F is a sheaf iff

(1) For each U, $F|_{U_{Zar}}$ is a Zariski sheaf

(2) $\{V\to U\}\in Cov(\mathcal{E})$, both V and U affine, the sequence

$0\to F(U)\to F(V)\to F(V\times_U V)$ is exact.

pf. because étale morphism is always open.

Prop. Let $A\to B$ f.f. Then $0\to A\to B\to B\otimes_A B\to B\otimes_A B\otimes_A B$ (**) comes from simplicial $A\to B\rightrightarrows B\otimes_A B\cdots$ is exact. Moreover, if M is an A-mod., then $M\otimes(**)$ is also exact.

pf. tensoring B over A, we get retraction $A\leftarrow B$, so use permanence, we are done.

Prop. $A\to B$ f.f., Z any scheme, then

$Hom(Spec\,A, Z)\longrightarrow Hom(Spec\,B, Z)\rightrightarrows Hom(Spec\,B\otimes_A B, Z)$ is an equalizer.

pf. use prop. before. (& the lemma before)

Thm. Let Z be a scheme, the presheaf $(Sch/X)^{op}_{fppf}\longrightarrow Set$

$Y\longmapsto Hom(Y,Z)$

is a sheaf for fppf top.

(Also, for étale top.)

Thm. Let F q.coh. on X. For $Y\xrightarrow{f}X$, set $F(Y)=f^*F$

Then the presheaf $(Sch/X)^{op}_{fppf}\to Ab$ is a sheaf on

$Y\longmapsto f^*F(Y)$

fppf site.

pf. Use the prop. and lemma before.

Thm. (Hilbert 90). The natural map $H^1_{Zar}(X,\mathbb{G}_m)\longrightarrow H^1_{et}(X,\mathbb{G}_m)$ is an isom.

$(\longrightarrow H^1_{fppf}(X,\mathbb{G}_m))$

pf. an element $s\in\check{H}^1_{et}(X,\mathbb{G}_m)$ is $\{U_i\to U\}$.

$s_{ij}\in\Gamma(U_i\times_U U_j,\mathcal{O}^*)$.

For each affine, we see s_{ij} ~~give~~ gives descent datum, ~~to~~ and by Permanence of being ~~free~~ l.f. rk 1, we see every étale line bdle must ~~be~~ come from a Zariski line bdle.

Defn. Given a sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)_{\acute{e}t}$, define $A_{\mathcal{F}} = \operatorname{colim}_{L/K} \mathcal{F}(L)$.

Defn. Given a discrete $G(k^{sep}/k)$-mod. A, define

$$\mathcal{F}(\coprod_i \mathrm{Spec}\, L_i) = \prod A^{G(k^{sep}/L_i)}$$

These give an equivalence $\mathrm{Sh}(\mathrm{Spec}(k)_{\acute{e}t}) \rightleftarrows \overset{\text{disc.}}{G(k^{sep}/k)\text{-mod}}$.

Cor. $H^i_{\acute{e}t}(\mathrm{Spec}(k), \mathcal{F}) = H^i_{Gal}(k, A_{\mathcal{F}})$.

Cohomology of curves

Set up: $k = \bar{k}$. X sm. curve conn. $/k$, $n \in \mathbb{Z}$, $\mathrm{char}\, k \nmid n$.

Recall: $\mathbb{G}_m$, sheaf of inv. functions. define $n: \mathbb{G}_m \to \mathbb{G}_m$, kernel μ_n.

Thm: $0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ is exact.

pf. $U \to X$ ét, $a \in \Gamma(U, \mathcal{O}_U^\times)$, need $V \to U$ étale, st. $\exists\, b \in \Gamma(V, \mathcal{O}_V^\times)$, $a = b^n$. Take $V = \underline{\mathrm{Spec}}(\mathcal{O}_U(T)/(T^n - a))$.

Thm ~~Thm~~: X proj. $H^i(X, \mu_n) = \begin{cases} \mu_n(k) & i=0 \\ \mathrm{Jac}(X)[n] & i=1. \\ \mathbb{Z}/n\mathbb{Z} & i=2 \\ 0 & \text{otherwise} \end{cases}$

X affine $H^i(X, \mu_n) = \begin{cases} \mu_n(k) & i=0 \\ \text{finite} & i=1 \\ 0 & i \geq 2. \end{cases}$

Computation of $H^i(X, \mathbb{G}_m)$.

$H^0(X, \mathbb{G}_m) = k^\times$ or $\mathcal{O}_X^\times$. $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$

Let R_X be sheaf of rat'l fctns $R_X(U) = K(U)$.

D_X be sheaf of ~~divisors~~ cartier divisors.

(?) Lemma 1. R_X & D_X are indeed sheaves.

Lemma 2. $0 \to \mathbb{G}_m \to R_X \to D_X \to 0$.

Lemma 3. $i: \mathrm{Spec}\, k(x) \to X$ be the inclusion. Then $R_X = i_* \mathbb{G}_{m, \mathrm{Spec}\, k(x)}$

pf. $i_* \mathbb{G}_m = (U \mapsto \mathbb{G}_m(U \times_X \mathrm{Spec}\, k(X)))$

$= (U \mapsto \Gamma(U, \mathcal{O}^\times_{U \times_X \mathrm{Spec}\, k(x)}))$

$= (U \mapsto k(U)^\times) = R_X(U)$.

Lemma 4. $D_X = \bigoplus_{x \text{ codim } 1} i_{x,*}(\mathbb{Z})$ if X locally factorial.

Lemma 5. $R^k i_* \mathbb{G}_m = 0$, $\forall\, i \geq 1$.

pf. $R^k i_* \mathbb{G}_m = (U \mapsto H^i(U \times_X \mathrm{Spec}(k(x))))^\#$

$= \bigoplus H^i(G_{L/k(x)}, L^\times)$

Tsen's thm $\Rightarrow H^i(G_{L/k(x)}, L^\times) = 0 \quad \forall\, L/k(x)$ finite, sep.

(?) Lemma 6. $H^i(X, x_* \mathbb{Z}) = 0 \quad \forall\, i \geq 1$.

$0 \to \mathbb{G}_m \to R_X \to D_X \to 0$.

$0 \to \mathbb{G}_m \to i_* \mathbb{G}_{m, \mathrm{Spec}\, k(x)} \to \bigoplus_{x \text{ closed}} x_*(\mathbb{Z}) \to 0$.

$H^i(X, R_X) = H^i(\operatorname{Spec} k(X), \mathbb{G}_m) = 0$ (by Tsen)

So $H^n(X, \mathbb{G}_m) = 0 \quad \forall\, n \geq 2$. both affine & proj. curve case.

So X proj. $\Rightarrow H^2(X, \mu_n) = \operatorname{coker}(n : \operatorname{Pic}(X) \to \operatorname{Pic}(X)) = \mathbb{Z}/n\mathbb{Z}$

X affine $\Rightarrow H^2(X, \mu_n) = \operatorname{coker}(n : \operatorname{Pic}(X) \to \operatorname{Pic}(X)) = 0$.

(?) Now $\operatorname{cok}(n : H^0(X, \mathbb{G}_m) \to H^0(X, \mathbb{G}_m)) = \operatorname{cok}(n : \mathbb{Z}^{\oplus_{x \in \bar{X} \setminus X}} \to \mathbb{Z}^{\oplus_{\bar{X} \setminus X}})$ is finite.

$H^1(G_{k(X)}, k(X)^\times) = 0$ by Hilbert 90.

Defn. central simple alg. $/k$ field, is a k-alg. fin dim'l simple w/ center k. simple : no 2-sided ideals.

Fact 1: $\exists$ a division alg. $/k$, w center k, s.t. $A \simeq M_n(D)$.

2: If A CSA, $\exists\, L/K$ fin. sep. s.t. $A \otimes_K L \cong M_n(L)$ for some n.

Defn. 2 CSA's eq. if D in fact 1 is the same.

$\{CSA's/k\}/\sim \; = H^2(k) = \operatorname{Br}(k)$.

Let D division alg. $/k(X)$ w/ center $k(X)$, $\det : D \otimes_k L \to L^\times$

get a map $\det : D \hookrightarrow M_n(k^{sep}) \to (k^{sep})^\times$, so det. is given by homogeneous poly. in n^2 variables of deg n.

Def. K is C_1 if all homogeneous polynomial in n variables of deg $d < n$ have a solution.

Thm (Tsen) $k(X)$ is C_1 for X sm. conn. curve $/k = \bar{k}$.

pf. Let F be a hom. of deg d, in n var's, $n > d$.

View F as $F : H^0(X, \mathcal{O}(qH))^n \to H^0(X, \mathcal{O}(dqH))$ just alg. map.

LHS has $\dim = nq \deg H + n(1-g)$.

RHS has $\dim = dq \deg H + (1-g)$.

So we have $f_1, \dots, f_n \in H^0(X, \mathcal{O}(qH)) \hookrightarrow H^0(X, k(X)) = k(X)$

s.t. $F(f_1, \dots, f_n) = 0$.

So the n before is just 1 ($n^2 \leq n$). And the only 1 dim'l div. alg. $/k$ is k itself. Hence $H^2(k(X)) = 0$.

About divisors $D_X = \bigoplus_{\substack{x \in X \\ \text{codim } 1 \\ \xi \text{ gen. pt}}} i_{\xi,*}(\underline{\mathbb{Z}})$. because RHS evaluated at (U)

$= \bigoplus_{\xi} i_{\xi,*}(\underline{\mathbb{Z}})(U)$

$= \bigoplus_{\xi} \bigoplus_{\eta} i_{\eta,*}(\underline{\mathbb{Z}})$, where $\xi \times_X U = \coprod \eta$'s.

About cokernels

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d, "n"] & \mathbb{Z}^{\#(\bar{X} \setminus X)} \arrow[r] \arrow[d, "n"] & Q \arrow[r] \arrow[d, "n"] & 0 \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & \mathbb{Z}^{\#(\bar{X} \setminus X)} \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

by snake lemma, we still have $\operatorname{coker}(n)$ is finite.

Étale cohomology Theory.

Henselian ring. (Fu Lei, Chap 2. §8.)

Prop & Defn (R, m) local, $k = R/m$ residue field, $S = \operatorname{Spec} R$, s closed pt, TFAE.

① finite R alg. $A = \prod A_{n_i}$, where A_{n_i} local rings.

② $\forall$ finite R alg. A, $A \to A/mA$ induces 1-1 correspondence between sets of idempotents (of monics)

③ primary decomposition in $R/mR[t]$ can be lifted to $R[t]$ uniquely.

④ simple roots of monics in R/mR can be lifted to R uniquely.

⑤ $\forall$ étale map $X \xrightarrow{g} S$, any section of $g_s : X \otimes_R k \to s$ is induced by a section of g.

In these cases we say R is henselian. If $k = R/m$ sep. closed, we say R is strictly henselian.

Prop. (R, m) complete local noetherian ring $\Rightarrow$ R henselian.

~~Defn~~ Prop. (A, m) local ring.

① A^h is a henselian local ring, $A \to A^h$ is local and faithfully flat, mA^h is the max. ideal of A^h, $A/m \xrightarrow{\sim} A^h/mA^h$.

② $\forall$ local henselian R, loc. $\operatorname{Hom}(A^h, R) \xrightarrow{\sim}$ loc. $\operatorname{Hom}(A, R)$.

③ If A is henselian, then $A \simeq A^h$.

④ $\hat{A} \xrightarrow{\sim} \hat{A^h}$

⑤ A noetherian $\Rightarrow$ A^h is noetherian.

Prop. A is reduced (or reg. ; normal) iff A^h is so iff A_i^{hs} is so

Prop $(R_\lambda, \varphi_{\lambda\mu})$. direct system of local rings, then

$$\left(\varinjlim_\lambda R_\lambda\right)^h = \varinjlim_\lambda R_\lambda^h. \quad \left(\varinjlim_\lambda R_\lambda\right)_i^{hs} \simeq \varinjlim (R_\lambda)_{i_\lambda}^{hs}.$$

Prop. (A, m) local, B finite A-alg., $n_1, \cdots, n_k$ max. ideals of B.
Then $B \otimes_A A^h \cong (B_{n_1})^h \times \cdots \times (B_{n_k})^h$. $(B \otimes_A A^{hs})_{\tilde{n}} = (B_n)^{hs}$

Prop. R henselian local ring, $S = \operatorname{Spec} R$, s closed pt. Any smooth morphism $X \xrightarrow{g} S$, section of $g_s : X_s \to \operatorname{Spec} k(s)$ can be lifted to a section of g.

Prop. Let (R, m) local, $k = R/m$ residue field, $S = \operatorname{Spec} R$, s closed pt.
TFAE. ① R strictly henselian
② R henselian, and any finite étale S-schem $X \simeq \coprod S$
③ $\forall$ étale $g : X \to S$ and any point $X \ni x$ lying above s, $\exists$ section $h : S \to X$, s.t. $h(s) = x$.

Prop. (A, m) local, $i : k = A/m \to \Omega = \bar{k}$. Then

① A_i^{hs} is shenselian local, $A \to A_i^{hs}$ is local, f.f., mA_i^{hs} is max'l, A_i^{hs}/mA_i^{hs} is k-isom. to Ω.

② R henselian local w/ res. field $k(R)$, $\varphi : A \to R$ local, and $\alpha : \Omega \to k(R)$ cptible w/ φ. $\exists!$ local A-hom. $\varphi' : A_i^{hs} \to R$ inducing α.

③ A s.h. $\Rightarrow A \simeq A_i^{hs}$. A noeth. $\Rightarrow A_i^{hs}$ noeth.

④ $i' : k \to \Omega'$, then $\forall$ k-isom $\sigma : \Omega \to \Omega'$, $A_i^{hs} \to A_{i'}^{hs}$ in ② is isom'c.
$\operatorname{Aut}(A_i^{hs}/A) = \operatorname{Gal}(\Omega/k)$.

Thm of proper base change.

Thm $X \to S$ proper, F sheaf of torsion abelian gps on X. Then, $\forall q \geq 0$, $R^q f_* F|_{\bar s}$ for $\bar s$ geom. pt $\simeq H^q(X_{\bar s}, F)$.

proof will be decomposed into 3 steps. (notice we may replace S by $\mathrm{Spec}(A)$, where A is local, s.h.).

0. Observe: Thm holds for finite morphisms. This follows from facts:
~~finite~~ A s.h., then $H^i(A, F) = 0$ $\forall i \geq 1$.
finite A-alg. will be product of s.h. alg's, # of product $\leq \dim_k B/m_B$

1. Use Chow's Lemma ~~to reduce~~ and induction on dimension of support and dévissage and blow up to reduce to $X \xrightarrow{proj.} S$, w/ fibre dim ≤ 1.

pf. Because we can ① $0 \to K \to F \to \pi_* \pi^* F \to C \to 0$.
② decompose $X \xrightarrow{proj.} S$ to $Bl\,X \to \mathbb{P}^1 \to S$.

Lemma
$$\begin{tikzcd} & \tilde X \arrow[dl, "\pi"'] \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$
① If thm holds for π, f, it holds for g, F.
② If π is finite, thm holds for g, it holds for $f, \pi_* F$.

pf. ①: $R^i f_* (R^j \pi_* F)|_{\bar s} \Longrightarrow R^{i+j} g_*|_{\bar s}$
$\wr\!\wr$
$H^i(X_s, R^j \pi_* F) \Longrightarrow H^{i+j}(\tilde X_s, F)$.

②: in this case $R^i \pi_* F = \begin{cases} \pi_* F & i = 0 \\ 0 & i \geq 1 \end{cases}$ □

2. To prove thm, for arbitrary torsion sheaf, we use an argument of limit to reduce to ~~locally~~ constant sheaves.

Defn & Prop. ~~TFAE~~ F sheaf on X, TFAE
① F is represented by finite étale $Y \to X$.
② $\exists$ covering $\{U_i \xrightarrow{f_i} X\}$, s.t. $F|_{U_i} \simeq \mathbb{Z}/n\mathbb{Z}$.
In this case, we call F locally constant.

pf. ① $\Rightarrow$ ②, structure thm for étale morphism: locally $X \times \mathbb{Z}/n\mathbb{Z}$.
② $\Rightarrow$ ①, locally $U_i \times \mathbb{Z}/n\mathbb{Z}$, descent to glue $Y \to X$.

Defn. F on X is constructible if $\exists$ stratification $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n$ st. $F|_{X_i - X_{i+1}}$ is locally constant.

Prop. cat. of constructible sheaves is an abelian cat. $F \xrightarrow{u} G$ is a hom of sheaves, F constructible, then $\mathrm{im}(F)$ is also constructible.

Lemma. Any torsion sheaf is filtered colimit of constructible sheaves.

pf. $\forall \xi \in F(U)$, we get $j_! \mathbb{Z}/n\mathbb{Z}_U \to F$. Image $\triangleq F_\xi$.
We see $F = \bigcup F_\xi$.

Defn. $\mathcal{C}$ abel. cat, T a functor $\mathcal{C} \xrightarrow{T} ab.$, is called effaceable if $\forall C \in Obj(\mathcal{C})$, and $\alpha \in T(C)$ $\exists$ $C \xhookrightarrow{u} M$, mono, s.t. $T(u)(\alpha) = 0$.

Lemma. $H^q(X, -)$ is effaceable on cat. of constructible sheaves.

pf. $F \hookrightarrow \prod_{x \in |X|} i_{\bar x *} F_{\bar x}$.

Lemma. $\varphi: T \to T'$ morph. of cohomological functors defined on an abel. cat $\mathcal{C}$, valued at Ab. If T^q is effaceable, $\forall q>0$. $\mathcal{E}$ a sub-collection of obj.$(\mathcal{C})$, and $\forall\, C \in Ob(\mathcal{C})$, $\exists\, C \hookrightarrow E \in \mathcal{E}$. TFAE

① $\varphi^q(A)$ bij. $\forall q \geq 0$ $\forall A \in Ob(\mathcal{C})$.

② $\varphi^0(M)$ is bij. and $\varphi^q(M)$ surj. $\forall q>0, M \in \mathcal{E}$.

③ φ^0 is bij $\forall A \in Ob(\mathcal{C})$ and T'^q is effaceable $\forall q>0$.

Prop. $X_0 \hookrightarrow X$ a subscheme. If $\forall\, n, q \geq 0$ and $\forall\, X' \to X$ finite,
$H^q(X', \mathbb{Z}/n\mathbb{Z}) \to H^q(X_0', \mathbb{Z}/n\mathbb{Z})$ is bij when $q=0$ and surj. when $q>0$. Then $\forall$ torsion sheaf on X, $\forall q \geq 0$.
$$H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X_0, \mathcal{F}).$$

pf. By passing to colimit, we may assume $\mathcal{F}$ constructible.
Using Lemma above, $T = H^\cdot(X, -)$, $T' = H^\cdot(X_0, -)$, $\mathcal{E}$ collection of constructible sheaf of the form $\prod p_{i*} C_i$, where $p_i: X_i' \to X$ finite, C_i constant sheaf on X_i'.

3. Compute the cohomology.

Prop. (A, m) henselian local, $X \xrightarrow{f} \mathrm{Spec}(A)$ proper, $X_0 = X \times_A A/m$. Then $\pi_0(X) \simeq \pi_0(X_0)$.

pf. 'long exact sequence of homotopy gps'.
Or: $\mathrm{Idem}\,\Gamma(X, \mathcal{O}_X) \underset{①}{\xrightarrow{\sim}} \mathrm{Idem}\,\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) \xrightarrow{\sim} \varprojlim \mathrm{Idem}\,\Gamma(X_n, \mathcal{O}_{X_n})$
top. inv. $\xrightarrow{\simeq} \mathrm{Idem}\,\Gamma(X_0, \mathcal{O}_{X_0})$.
①: by henselian and proper   Thm of formal fcns.

Prop. $H^1(X, \mathbb{Z}/n\mathbb{Z}) \simeq \{Y \to X,$ ~~[illegible]~~ Galois covering w/ $\mathrm{Gal}(Y/X) = \mathbb{Z}/n\mathbb{Z}\}$

pf. $\Leftarrow$, use structure thm to give cocycles
$\Rightarrow$ construct Y locally ~~by gi~~ by product, glue by descent.

Prop. A henselian local, $X \xrightarrow{f} \mathrm{Spec}(A)$ proper w/ fibre X_0. Then we have $\mathrm{FÉt}_X \xrightarrow{\sim} \mathrm{FÉt}_{X_0}$.

~~Prop. $0 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{\ell \cdot x} \mathbb{G}_m \to 0$ is exact. So~~
~~So $\forall \ell \geq 0$, $H^q(X_0, \mathbb{Z}/p^\ell\mathbb{Z}) = 0$, $\forall X_0$~~

If $p \nmid n$, we have
$$\begin{array}{ccc} \mathrm{Pic}(X) & \longrightarrow & H^2(X, \mathbb{Z}/n\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X_0) & \longrightarrow & H^2(X_0, \mathbb{Z}/n\mathbb{Z}) \end{array}$$

Prop. $X \to S$, S henselian local, X_0 of dim ≤ 1, Then $\mathrm{Pic}(X) \twoheadrightarrow \mathrm{Pic}(X_0)$.

pf. $0 \to I \to \mathcal{O}^\times_{X_n} \to \mathcal{O}^\times_{X_{n-1}} \to 0$, So $\mathcal{L}_0$ on X_0 can be lifted to $\mathfrak{X} = \hat{X}_{0,m}$, by Grothendieck's Existence Thm.
$\mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots \leftarrow \hat{\mathcal{L}}$
Then by approximation, we see $\exists\, X \to \tilde{X} \hookleftarrow \mathfrak{X}$.

$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow \mathfrak{X}$

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \tilde{\mathcal{L}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \tilde{X} \end{array}$$

so pull $\hat{\mathcal{L}}$ to get $\mathcal{L}$.

Defn. X sep. finite type over field k. By a thm of Nagata, $\exists\, X \overset{i}{\hookrightarrow} \bar{X}$ over k, where i is an open immersion, $\bar{X}$ proper over k.

$H^q_c(X,F) = H^q(\bar{X}, i_! F)$.

Lemma.
$$\begin{tikzcd} X \arrow[r, "j_2"] \arrow[dr, "j_1"'] & \bar{X}_2 \arrow[d, "p"] \\ & \bar{X}_1 \end{tikzcd}$$
Then $p_*(j_{2!}F) = j_{1!}F$, $R^q p_*(j_{2!}F) = 0$, $\forall q > 0$.

pf. compute stalks

Similarly $X \to S$, sep. finite type of Noetherian schemes, $\exists$
$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr] & \bar{X} \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$
define $R^q f_! F = R^q \bar{f}_*(j_! F)$.

Thm.
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$
we have $g^*(R^q f_! F) \simeq R^q f'_!(g'^* F)$, where F is a torsion sheaf.

Thm $f: X \to S$ sep. finite type, fibre dim $\le n$, F a torsion sheaf. Then $R^q f_! F = 0$, $\forall q > 2n$.

Thm $f: X \to S$, where both X, S sep finite type over $\mathbb{C}$. Then $(R^q f_! F)^{an} \cong R^q f_!^{an} F^{an}$. In particular $H^q_c(X, \mathbb{Z}/n\mathbb{Z}) \simeq H^q_c(X^{an}, \mathbb{Z}/n\mathbb{Z})$.

pf. by dévissage ~~and other~~, reduce to X: is a smooth curve over a pt, $F = \mathbb{Z}/n\mathbb{Z}$. Then ~~By Ei~~ $\pi_0(X) = \pi_0(X^{an})$ and $\pi_1(X) = \pi_1(X^{an})$. And by GAGA $\mathrm{Pic}(X) = \mathrm{Pic}(X^{an})$. So proved for $H^{0,1,2}$.

Thm X affine scheme of finite type over $k = k^{sep}$, F a torsion sheaf on X. Then $H^q(X, F)$ for $q > \dim X$.

$H^2(X, \mathbb{Z}/p\mathbb{Z})$ for X curve, $p = \mathrm{char}(k)$

$0 \to \mathbb{F}_p \to \mathbb{G}_a \xrightarrow{1-F} \mathbb{G}_a \to 0$.

$H^1(X, \mathbb{G}_a) = H^1(X, \mathcal{O}_X)$

$V \xrightarrow{F} V$. Let $W = \ker(F^\infty)$.

Then $V/W \xrightarrow{\sim} V/W$. So
$$\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r, "F"] \arrow[dr, "\mathrm{Frob}"'] & \mathbb{P}^{n-1} \\ & \mathbb{P}^{n-1} \end{tikzcd}$$

~~Pf~~ By Lefschetz trace formula + calculation ...

(or Abelian Varieties P143.)

Thm. $X \xrightarrow{f} S$ sep. finite type, $R^q f_!$ send constructible to constructible.

pf. ... really hard ?!!

Smooth base change thm.

$Y \xrightarrow{\pi} X$, π is acyclic if $\forall$ all $X' \to X$ finite étale $\mathcal{F}$ torsion prime to char(X) on X', we have
$$H^i(X', \mathcal{F}) \simeq H^i(Y \times_X X', \pi'^* \mathcal{F}).$$

Defn. π is universally acyclic if $\forall$ $X' \to X$, π' is acyclic.

Defn. π is (universally) locally acyclic if $\forall$ geom. pt $\bar{y}$ of Y, the map $\mathrm{Spec}(\mathcal{O}^{sh}_{Y,\bar{y}}) \longrightarrow \mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{y}})$ is (universally acyclic).

Thm.
$$\begin{tikzcd} Y \arrow[d, "\pi"'] & Y' \arrow[l, "g'"'] \arrow[d, "\pi'"] \\ X & X' \arrow[l, "g"] \end{tikzcd}$$
π is q-cpt, g is smooth. $\mathcal{F}$ torsion sheaf on Y (prime to char(X)).

Then $g^*(R^i\pi_*\mathcal{F}) \simeq R^i\pi'_*(g'^*\mathcal{F})$ $\forall i$.

pf. by structure thm, reduce to $X = \mathrm{Spec}(A)$, where A strictly henselian local ring, $X' = \mathrm{Spec}(A\{T\})$, where $A\{T\} = A[T]^{sh}_{(m_A, T)}$, and $H^i(X, \mathbb{Z}/\ell\mathbb{Z}) = H^i(X \times_A A\{T\}, \mathbb{Z}/\ell\mathbb{Z})$.

This follows from the following Lemma.

Lemma. (acyclicity) $A \xrightarrow{g} B$ smooth morphism between s.h. local rings, (say, $A \to A\{T\}$), then $\forall$ $\mathcal{F}$ torsion prime to ~~char(A)~~ char(A/m_A), then we have

(a) $\mathcal{F} \simeq g_* g^* \mathcal{F}$

(b) $R^i g_*(g^*\mathcal{F}) = 0$ $\forall i > 0$.

To see lemma applies thm, we just saw that it suffices to prove for étale morphism ~~$Y \to X$~~ $X \xrightarrow{\pi} \mathrm{Spec}(A)$ (as q-proj. will be proper $\circ$ open immersion, and by means of s.s. for affine covering we reduce to $X \to \mathrm{Spec}(A)$ affine (which is proj-lim of f.g. affine (so q-proj.))).

And for étale morphism $X \xrightarrow{\pi} \mathrm{Spec}(A)$, we see
$$H^i(X, \mathbb{Z}/\ell\mathbb{Z}) = H^i(X, g_*g^*\mathbb{Z}/\ell\mathbb{Z}) = H^i(X \times_A A\{T\}, \mathbb{Z}/\ell\mathbb{Z}).$$

Application Thm. $f: X \to S$ proper and locally acyclic, say, smooth proper. Then $R^q f_* \mathbb{Z}/n\mathbb{Z}$ is locally constant constructible and $\forall$ $t \to \tilde{S}^s$ specialization, we have $H^i(X_t, \mathbb{Z}/n\mathbb{Z}) \simeq H^i(X_s, \mathbb{Z}/n\mathbb{Z})$ where n is invertible on char(res(S)).

Cor. for $K \supseteq k$, both sep. closed, X/k proper smooth, $(n, \mathrm{char}(k)) = 1$. Then $H^q(X, \mathbb{Z}/n\mathbb{Z}) \simeq H^q(X_K, \mathbb{Z}/n\mathbb{Z})$ $\forall q \geq 0$.

pf. K is inductive limit of smooth k-alg.'s

Thm. (relative purity)
$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr] & X \arrow[d, "f"] & Y \arrow[l, hook', "i"'] \arrow[dl, "h"] \\ & S & \end{tikzcd}$$
f smooth rel dim N, $Y \xhookrightarrow{i} X$ closed imm. $Y \xrightarrow{h} S$ rel smooth dim $N-1$. $U = X \setminus Y$.

Then $j_* \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n\mathbb{Z}$.
$$R^1 j_* \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n\mathbb{Z}(-1)_Y.$$
$$R^q j_* \mathbb{Z}/n\mathbb{Z} = 0 \quad \text{for } q \geq 2.$$

May assume
$$\begin{tikzcd} \mathbb{A}^1_T \arrow[r, hook] \arrow[dr] & \mathbb{P}^1_T \arrow[d] & T \arrow[l, hook'] \arrow[dl] \\ & T & \end{tikzcd}$$
and run Leray S.S. for

about locally constancy of (coh. of) sm. proper family:

$$\begin{tikzcd} X_{\bar\eta} \arrow[r, "j"] \arrow[d] & X \arrow[d] \\ \bar\eta \arrow[r] & A \end{tikzcd} \quad F$$

(by proper base change) reduce to prove $F \simeq j_* j^* F$, $R^i j_*(j^* F) = 0$ for $i > 0$

but $X \to A$ is smooth, so ~~...~~ locally (universally) acyclic.

Hence we are done.

Thm (Grothendieck) Suppose $X = \varprojlim X_\alpha$, then we have $X_{ét} = \varinjlim X_{\alpha, ét}$ in the sense of étale site.

Fact $X \to Y$ reasonably nice spaces, étale morphism.
$\exists$ stratification on Y, s.t. on each strata, $X_i \to Y_i$ is finite étale.

Poincaré duality: $X/k=\bar k$.

$$H^p(X, \mu_n^{\otimes d}) \times H_c^{2d-p}(X, \mathbb{Z}/n\mathbb{Z}) \longrightarrow H_c^{2d}(X, \mu_n^{\otimes d}) \xrightarrow{Tr} \mathbb{Z}/n\mathbb{Z}.$$

We can define trace map and it'll be an isomorphism.

Refn. 1° a morphism $f: X \to Y$ is called of type S if it is smooth, rel. dim d, compactified over Y.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & \bar X \arrow[ur, "\bar f"'] & \end{tikzcd}$$

$H_c^i(X, F) \triangleq H^i(\bar X, j_! F)$

$R^i f_!: D^+(X, tor) \longrightarrow D^+(Y, tor)$.
$R^i f_! = R^i \bar f_* \circ j_!$

For any $X \to Y$ type S, define a trace map
$tr: R^{2d} f_! \mu_n^{\otimes d} \longrightarrow \mathbb{Z}/n = \Lambda$.
$\mu_n^{\otimes d} = T_{X/Y}$

1° étale morphism, $f_! f^* \Lambda \longrightarrow \Lambda$, gives $f_! \Lambda \longrightarrow \Lambda$.

2° $X/k=\bar k$, curve.

$$\begin{tikzcd} H_c^2(X, \mu_n) \arrow[r] \arrow[d] & \mathbb{Z}/n\mathbb{Z} \\ H^2(\bar X, \mu_n) \arrow[ur] & \end{tikzcd}$$

$X \overset{j}{\hookrightarrow} \bar X$

$0 \to j_! \mu \to \mu \to i_* \mu \to 0$.

3° Given $X \xrightarrow{g} T \xrightarrow{h} S$

for h: $R^{2d_{T/S}} h_! T_{T/S} \longrightarrow \Lambda_S$

for g: $R^{2d_{X/T}} g_! T_{X/T} \longrightarrow \Lambda_T$.

so use S.S. $\Rightarrow$ trace map well defined for $h \circ g$.

cup product: $X \hookrightarrow \bar{X}$ $\quad \mathcal{F}, \mathcal{G}$

$j_!\mathcal{F} \to I^\cdot \quad j_!\mathcal{G} \to J^\cdot$.

$R\mathrm{Hom}(\mathcal{F},\mathcal{G}) = \mathrm{Hom}(\mathcal{F}, j^*J^\cdot) = \mathrm{Hom}(j_!\mathcal{F}, J^\cdot)$

$= \mathrm{Hom}(I^\cdot, J^\cdot) \longrightarrow \mathrm{Hom}(\Gamma_c(\bar{X}, I^\cdot), \Gamma_c(\bar{X}, J^\cdot))$

So we get $\mathrm{Ext}^p(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{Hom}(H^q_c(X,\mathcal{F}), H^{p+q}_c(X,\mathcal{G}))$

Or:

$$\mathrm{Ext}^p(\mathcal{F},\mathcal{G}) \times H^q_c(X,\mathcal{F}) \longrightarrow H^{p+q}_c(X,\mathcal{G}).$$

take $\mathcal{G} = T_X$, $p+q=2d$.

$$\begin{array}{ccc} \mathrm{Ext}^p(\mathcal{F}, T_X) & \longrightarrow & \mathrm{Hom}(H_c^{2d-p}(X,\mathcal{F}), H^{2d}_c(X,T_X)) \\ & & \downarrow \mathrm{Tr} \\ & & \mathbb{Z}/n\mathbb{Z} \end{array}$$

To prove it's isom. we just have to find an collection $\mathcal{M}$ in Constructable(X), s.t.

① $F_1^0(M) = F_2^0(M)$ $\forall M \in \mathcal{M}$.

② $\forall A \in \mathcal{C}(X)$, $a \in F_i^k(A)$, $\exists M_i^k \xrightarrow{f_i^k} A$, s.t. $F_i^k(f_i^k)(a) = 0$.

And we get what we want in the end of the day.

For curve: branched covering trick.

in general: dévissage.

Trace Formula

X finite type $/k$, separated.

Thm $\quad \sum_{x \in X^{\mathrm{Frob}^d}} \mathrm{Tr}_\Lambda(\mathrm{Fr}^d, K_x) = \mathrm{Tr}_\Lambda(\mathrm{Fr}^d \mid R\Gamma(X, K^\cdot))$.

where $K \in D_{perf}(X, \Lambda)$. K locally quasi-isom. to bdd cplx whose cohomology are constructible f.g. flat $/\Lambda$.

Thm (Weil) $\quad X$ sm. proj. curve over $k=\bar{k}$. $\varphi: C \to C$ non-constant morphism, then in $C \times C$, $(\Gamma_\varphi, \Delta) = 1 - \mathrm{tr}_J(\varphi^*) + \deg\varphi$.

pf. $(\Gamma_\varphi, \Delta) = (\Gamma_\varphi, \Delta - \{p\} \times C - C \times \{p\})$

$\quad + 1 + \deg \varphi$

$\quad = \int_{C \times C} c_1(\Gamma_\varphi) \cdot \sigma^*(\mathrm{Corr}\,\xi) + 1 + \deg\varphi$.

Then we use the fact that if $a, b \in H^2(C \times C)$ corresponds to 2 correspondences $\in \mathrm{End}(J(C))$, then

$\mathrm{Tr}_J(ab) = -\int_{C\times C} a \cdot \sigma^* b$.

So RHS $= 1 - \mathrm{Tr}_J(\varphi^*) + \deg\varphi$.

To prove Trace Formula, do the dimension 0, 1 cases and then dévissage. 0-dim'l ~~are~~ is easy.

1-dim'l, by hard alg. reduce to C smooth affine, $\mathcal{F}$ constant, then we have it as above.

1. Recall. $X_0/\mathbb{F}_q$, smooth, proj., geom. ~~connected~~ irreducible, we defined $Z(X_0,t)=\prod_{x\in|X|}(1-t^{\deg(x)})^{-1}$

and we find out $Z(X,t)=\dfrac{\prod_{i \text{ odd},\, i=1}^{2n-1} P_i(t)}{\prod_{i \text{ even},\, i\geq 0}^{2n} P_i(t)}$,

where $P_i(t)=\det(1-F^*t, H^i_c(\bar{X},\mathbb{Q}_\ell))^{(-1)^{i+1}}$

Defn. a number z is called a Weil number of weight i if $z\in\bar{\mathbb{Q}}$ and $\forall\ \bar{\mathbb{Q}}\xrightarrow{\varphi}\mathbb{C}$,

$|\varphi(z)|=q^{-i/2}$

Thm (Deligne) all the roots of $P_i(t)$ are Weil numbers of weight i $(W(X_0,i))$.

$H^i_c(H^i)(X,\mathbb{Q}_\ell)=\varprojlim_n H^i_c(H^i)(X,\mathbb{Z}/\ell^n\mathbb{Z})\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$.

Observing ① $X_{\mathbb{F}_q}\times\mathbb{F}_{q^r}/\mathbb{F}_{q^r}\Leftrightarrow X/\mathbb{F}_q$

Thinking: ② What if we know thm for $Bl_Z X$?

$H^i(X)\hookrightarrow H^i(\tilde{X})$. we know that of X!

③ What if we have a smooth proj fibration: $X\xrightarrow{f}Y$.

Then we get $E_2^{p,q}=H^p(Y,R^qf_*\mathbb{Q}_\ell)\Rightarrow H^{p+q}(X,\mathbb{Q}_\ell)$.

And it suffices to show thm holds for $E_2^{p,q}$...

④ weak Lefschetz + Poincaré duality $\Rightarrow$ only interested in middle cohomology.

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2. Lefschetz pencil & Monodromy.

We want to control the geometry of the fibration $X\xrightarrow{f}Y$.

Thm. we may choose an embedding $X\hookrightarrow\mathbb{P}^n_{\mathbb{F}_q}$, and choose a codim 2 plane in $\mathbb{P}^n$: A, s.t. $\tilde{X}\subseteq X\times\mathbb{P}^{1\vee}$

$\tilde{X}=\{(x,H)\mid A\subseteq H,\ x\in H\cap X\}=Bl_{A\cap X}X$

several Frobenii:

$$\begin{tikzcd} {}_k^{q}X^{q} & {}_kX \arrow[l, "\text{absolute Frobenius}"'] \arrow[d, "\text{geom. Frobenius}"] \\ & {}_kX^{q} \arrow[ul, "\text{arithmetic Frobenius}"] \end{tikzcd}$$

k finite $\mathbb{F}_q$, $q=p^r$

$$\begin{tikzcd} X \arrow[rr, "F_X^r"] \arrow[dr] & & X \arrow[dl] \\ & k & \end{tikzcd}$$

$$\begin{tikzcd} \bar{X} \arrow[rr, "F_X^r\times 1_{\bar{k}}"] \arrow[dr] & & \bar{X} \arrow[dl] \\ & \bar{k} & \end{tikzcd}$$

geom. Frob.

$\sigma\in Gal(\bar{k}/k)$.

$$\begin{tikzcd} \bar{X} \arrow[r, "1_X\times\sigma"] \arrow[d] & \bar{X} \arrow[d] \\ \bar{k} \arrow[r, "\sigma"] & \bar{k} \end{tikzcd}$$

arithmetic Frob.

② $\pi_1(\mathbb{P}^1-S,\omega)$ acts trivially on $R^{n+1}f_*\mathbb{Q}_\ell|_\omega$ and tamely on $R^n f_*\mathbb{Q}_\ell|_\omega$.

For $x\in R^n f_*\mathbb{Q}_\ell|_\omega$ and $\xi\in Z^{(p)}(1)\subseteq\mathbb{Q}_\ell(1)$; $\gamma_s(\omega)(x)=x-(-1)^m\xi\cdot\langle x,\delta_s\rangle\delta_s$.

Recall:

$$\begin{tikzcd} & \mathrm{Spec}(\mathcal{O}^{sh}_{\mathbb{P}^1,s})=\mathbb{P}^1(s) \arrow[d] \\ \mathrm{Spec}(\overline{\mathbb{F}_q(t)}) \arrow[ur, "\tilde\omega"] \arrow[r, "\omega"] & \mathbb{P}^1-S \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(\mathbb{P}^1(s)-\{s\},\tilde\omega) \arrow[r] \arrow[d, "\gamma_{\tilde\omega,s}"] & \pi_1(\mathbb{P}^1-S,\omega) \arrow[d] \\ \hat{Z}^{(p')}(1) \arrow[r, dashed] & \pi_1^t(\mathbb{P}^1-S,\omega) \end{tikzcd}$$

$$\gamma_{\tilde\omega,s}(\sigma)=\varprojlim_{p\nmid r}\frac{\sigma(\sqrt[r]{\pi})}{\sqrt[r]{\pi}}$$

$$\pi_1^t=\pi_1(\mathbb{P}^1-S,\omega)\Big/\text{normal closure of }\langle\ker(\gamma_{\tilde\omega,s})\rangle$$

So acting tamely simply means factor thru π_1^t, (locally factor thru $\gamma_{\tilde\omega,s}$)

③ all these δ_s's are conjugate to each other by $\pi_1(\mathbb{P}^1-S,\omega)$.

Cor. Case 1. vanishing cycles not 0:

① $\forall i\neq n$, $R^i f_*\mathbb{Q}_\ell$ are constant sheaves

② $j:\mathbb{P}^1-S\hookrightarrow\mathbb{P}^1$, then $R^n f_*\mathbb{Q}_\ell=j_*j^*R^n f_*\mathbb{Q}_\ell$.

③ Let $E=\sum\mathbb{Q}_\ell\delta_s\subseteq R^n f_*\mathbb{Q}_\ell|_\omega$, it's stable under $\pi_1(\mathbb{P}^1-S,\omega)$ action, and $E^\perp=(R^n f_*\mathbb{Q}_\ell|_\omega)^{\pi_1(\mathbb{P}^1-S,\omega)}$

$E/(E\cap E^\perp)$ is an irreducible rep'n of $\pi_1(\mathbb{P}^1-S,\omega)$.

s.t. $OA\cap X$ is smooth subvariety, hence $\tilde X$ remain smooth,

② $P_2:\tilde X\longrightarrow\mathbb{P}^{1\vee}$ is smooth outside of finitely many pts $x_i\in\tilde X$, and x_i belong to different fibers. (assume they are rat'l pts after base extn)

③ $\widehat{\mathcal{O}_{\tilde X,x_i}}\cong\mathbb{F}_q[[X_1,\cdots,X_n]]/(f)$, where $f\in m^2$, m: maximal ideal of $\mathbb{F}_q[[X_1,\cdots,X_n]]$, $f\equiv Q \bmod m^3$, and Q is a ~~nondeg~~ nonsingular quadratic form.

these x_i's are called ordinary double point (simplest singularity one would imagine?).

Now away from $S=\{f(x_i)=s_i\}$, we see $R^i f_*\mathbb{Q}_\ell$'s are locally constant sheaves by smooth & proper base change thms. Hence ~~they are naturally~~ we assign $R^i f_*\mathbb{Q}_\ell|_\omega=H^i(\tilde X_\omega,\mathbb{Q}_\ell)$ a $\pi_1(\mathbb{P}^1-S,\omega)$-module structure. where ω is ~~$\mathrm{Spec}(\overline{\mathbb{F}_q(t)})$ generic~~ geometric pt of $\mathbb{P}^1-S$.

Thm (Picard-Lefschetz formulas) Call $V=R^n f_*\mathbb{Q}_\ell|_\omega$, assume $\dim\tilde X=n+1$, $n=2m+1$

⓪ $R^i f_*\mathbb{Q}_\ell$ are locally constant for $i\neq n,n+1$, hence constant on $\mathbb{P}^1$, as $\mathbb{P}^1$ is simply connected.

① $\forall s\in S$, there is a "vanishing cycle" δ_s in $V(m)$ depends up to sign and conjugation only on s, $\delta_s^*\in R^{n+1}f_*\mathbb{Q}_\ell|_s(n-m)$

and an exact sequence

$$0\to R^n f_*\mathbb{Q}_\ell|_s\longrightarrow R^n f_*\mathbb{Q}_\ell|_\omega\longrightarrow\mathbb{Q}_\ell(m-n)\longrightarrow R^{n+1}f_*\mathbb{Q}_\ell|_s\longrightarrow R^{n+1}f_*\mathbb{Q}_\ell|_\omega\to 0$$

$$x\longmapsto(x,\delta_s)$$

$$y\longmapsto y\delta_s^*$$

2) $E_2^{0,n+1}$: If ~~E≠0~~ vanishing cycle $\neq 0$, then $R^{n+1}f_*\mathbb{Q}_\ell$ is constant,

and $E_2^{0,n+1} = H^{n+1}(X_u, \mathbb{Q}_\ell)$ and the Gysin map

$$H^{n-1}(Y, \mathbb{Q}_\ell)(-1) \twoheadrightarrow H^{n+1}(X_u, \mathbb{Q}_\ell) \ldots$$

If v.c. $= 0$, then ~~we~~ by ~~Lef~~ Picard-Lefschetz, we have

$$0 \to \bigoplus_{s\in S} \mathbb{Q}_\ell(m-n) \to E_2^{0,n+1} \to H^{n+1}(X_u, \mathbb{Q}_\ell) \to 0$$

$2m+1=n$, $m-n=-m-1=-\frac{d}{2}$. So F^* acts by $q^{\frac{d}{2}}$.

$H^{n+1}(X_u, \mathbb{Q}_\ell)$ is handled as above.

3). $E_2^{1,n} = H^1(\mathbb{P}^1, R^n f_*\mathbb{Q}_\ell)$, if v.c $=0$, $R^n f_*\mathbb{Q}_\ell$ is constant,

so no H^1...

If v.c $\neq 0$. Case 1: $\delta_s \notin \mathcal{E}^\perp$, then we have

$$0 \to j_*\mathcal{E} \to R^n f_*\mathbb{Q}_\ell \to \mathcal{G} = \text{some constant sheaf} \to 0$$

$$0 \to j_*(\mathcal{E}\cap\mathcal{E}^\perp) = \text{constant} \to j_*\mathcal{E} \to j_*(\mathcal{E}/(\mathcal{E}\cap\mathcal{E}^\perp)) \to 0.$$

~~we the~~

Key fact: Use Rankin's Method (§3 in Weil I) and a trick (§6 in W I)

we can prove that $H^1(\mathbb{P}^1, j_*(\mathcal{E}/\mathcal{E}\cap\mathcal{E}^\perp))$ satisfy the thm.

So then as constant sheaf has no H^1, we ~~see~~ are done.

Case 2: $\delta_s \in \mathcal{E}^\perp$, then $\mathcal{E} \subseteq \mathcal{E}^\perp$. So we have

$$0 \to j_*\mathcal{E}^\perp = \text{constant} \to R^n f_*\mathbb{Q}_\ell \to \mathcal{F} \to 0.$$

$$0 \to \mathcal{F} \to j_*j^*\mathcal{F} = \text{constant} \to \bigoplus_{s\in S} \mathbb{Q}_\ell(n-m)_s \to 0.$$

use F^* act on $\bigoplus_{s\in S}\mathbb{Q}_\ell(n-m)$ by $q^{d/2}$ and $H^1(\text{Const})=0$.

Case 2: Vanishing cycles are 0 (conjugate to each other).

① $\forall\, i \neq n+1$, $R^i f_*\mathbb{Q}_\ell$ are constant sheaves.

② $0 \to \bigoplus_{s\in S}\mathbb{Q}_\ell(m-n)_s \to R^{n+1}f_*\mathbb{Q}_\ell \to \mathcal{F} \to 0$ where $\mathcal{F}$ is a constant sheaf.

③ $E=0$.

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3. The proof.

Lemma 1. $X_0/\mathbb{F}_q$, even d-dim'l, geom. irreducible ~~connected~~ smooth proj. variety. $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, α is an eigenvalue of F^* on $H^d(X, \mathbb{Q}_\ell)$. Then $\alpha \in \overline{\mathbb{Q}}$, and $\forall\ \overline{\mathbb{Q}} \xrightarrow{\varphi} \mathbb{C}$,

$$q^{\frac{d}{2}-\frac{1}{2}} \le |\varphi(\alpha)| \le q^{\frac{d}{2}+\frac{1}{2}}$$

pf. ~~Proof~~ Suppose we have $X_0 \hookrightarrow \tilde{X}_0$, and $\tilde{X}_0 \xrightarrow{f_0} \mathbb{P}^1_0$ Lefschetz pencil, after base extn, every thing is defined $/\mathbb{F}_q$, Y_0 a hyperplane section of X_{u_0}, $u_0 \in \mathbb{P}^1_0 - S$, $u = u_0$

$$E_2^{p,q} = H^p(\mathbb{P}^1, R^q f_*\mathbb{Q}_\ell) \Rightarrow H^{p+q}(\tilde{X}, \mathbb{Q}_\ell).$$

Suffices to prove the statement for all $E_2^{p,q}$ $(p+q=d=n+1)$.

1) $E_2^{2,n-1} = H^2(\mathbb{P}^1, R^{n-1}f_*\mathbb{Q}_\ell) \cong H^0(\mathbb{P}^1, R^{n-1}f_*\mathbb{Q}_\ell^\vee(-1))$

(as $R^{n-1}f_*\mathbb{Q}_\ell$ is constant) $\cong H^{n-1}(\tilde{X}_u, \mathbb{Q}_\ell)(-1)$

Now by weak Lefschetz thm:

$$H^{n-1}(\tilde{X}_u, \mathbb{Q}_\ell)(-1) \hookrightarrow H^{n-1}(Y_u, \mathbb{Q}_\ell)(-1),$$

where Y is an hyperplane section of $\tilde{X}_u$ which has dim $d-2 = n-1$

use induction of Y_0.

Lemma 2. ~~Let~~ $X_0/\mathbb{F}_q$, assumption as before, ~~α is an~~ α is an eigenvalue of F^* on $H^d(X, \mathbb{Q}_\ell)$, then α is a Weil number of weight d.

pf. $\forall k$ even, α^k is an eigenvalue of F^* on $H^{kd}(X^k, \mathbb{Q}_\ell)$. so $q^{\frac{kd}{2}-\frac{1}{2}} \le |\varphi(\alpha)|^k \le q^{\frac{kd}{2}+\frac{1}{2}}$.

let $k \to \infty$, $|\varphi(\alpha)| = q^{\frac{d}{2}}$.

Now prove Deligne's Thm:

denote $W(X_0, i)$ the RH for $H^i(X, \mathbb{Q}_\ell)$. Then we see by Poincaré duality, it suffices to show $W(X_0, i)$ $\forall$ $i \le \dim X_0$.

for $i < n$, $W(X_0, i)$ will be implied by $W(Y_0, i)$ where Y_0 is a smooth hyperplane section, by weak Lefschetz.

for $i = n$, it's Lemma 2.

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4. Application: E_2 degeneration of Leray S.S. for ~~prope~~ proj sm. family w/ base simply connected

Rmk: this holds for ~~variety of the form $X_0 - Y_0$~~ ~~non~~ separable integral f.t. scheme$/\mathbb{F}_q$.

but we only have H^i_c has weight $\le i$.

And there will be a weight filtration.